Question Bank

Unit-1	Function of Bounded Variation
[A]	**1 – Mark Questions**
1.	Define Monotonically Increasing function.
2.	Define Monotonically Decreasing function.
3.	If f is monotonic on [a, b], then what you can conclude about countability for the set of discontinuities of f?
4.	What do you mean by partition of [a,b]?
5.	Define Bounded variation.
6.	Show that if f is monotonic on [a,b], then f is of bounded variation on [a,b].
7.	Define Total variation.
[B]	**3 – Marks Questions**
1.	Show that if f is of bounded variation on [a,b], say $\sum\lvert\Delta f_k\rvert \le M$ for all M for partitions of [a,b], then f is bounded on [a,b]. In fact, $\lvert f(x)\rvert \le \lvert f(a)\rvert + M$ for all x in [a,b].
2.	Give an example to show that boundedness of f' is not necessary for f to be of bounded variation.
3.	Assume that f and g are each of bounded variation on [a,b]. Then prove that their sum, difference and product are also bounded variation. Also we have $V_{f+g} \le V_f + V_g$ and $V_{f\cdot g} \le AV_f + BV_g$ where A=$\sup\{\lvert g(x)\rvert : x \in [a,b]\}, B = \sup\{\lvert f(x)\rvert : x \in [a,b]\}$.
4.	Determine whether the following function is bounded variation on [0,1] or not. $f(x) = x^2\sin(x/y)$ if $x \ne 0, y \ne 0, f(0) = 0$.
5.	Determine whether the following function is bounded variation on [0,1] or not. $f(x) = \sqrt{x}\sin(x/y)$ if $x \ne 0, y \ne 0, f(0) = 0$.
6.	State Mean Value Theorem and prove that if g is continuous on [a, b] and if f' exists and is bounded in the interval, say |f' (x)|≤ A for all x in (a,b) then f is bounded variation on [a,b].
7.	Define Bounded Variation and verify that $f(x) = x.cos\frac{\pi}{2x}\ if\ x \ne 0, f(0) = 0\ in\ [0,1]$ is bounded variation or not.
8.	Show that f be defined on [a,b] then f is of Bounded Variation on [a,b] if and only if f can be expressed as difference of two increasing functions.
9.	Determine whether the following function is bounded variation on [0,1] or not. $f(x) = x^2\cos(1/x)$ if $x \ne 0, f(0) = 0$.
[C]	**5 – Marks Questions**
1.	Let f be of bounded variation on [a,b], and assume that c in (a,b). Then f is of bounded variation on [a,c] and [c,d] and we have $V_f(a,b) = V_f(a,c) + V_f(c,b)$.
2.	State and prove additive property of total variation.
3.	Let f be of bounded variation on [a, b]. Let V be defined on [a, b] as follows: V(x) = Vf(a, x) if a < x ≤ b, V(a) = 0 then : i) V is an increasing function on [a, b] ii) V- f is an increasing function on [a, b].
4.	State and prove necessary and sufficient condition for bounded variations.

5.	Let f be a bounded variation on [a, b]. Shoe that if x∈ $(a, b]$, let V(x)= $V_f(a, x)$ and put V(a) = 0. Then every point of f is also a point of continuity of V.
Unit-2	**Metric Spaces**
[A]	**1 – Mark Questions**
1.	Define Distance of a point from a given set.
2.	Define Distance between two subsets of a metric space.
3.	Define Diameter of a subset of metric space.
4.	What do you mean by bounded metric space?
5.	Write whether the following statement is True or False with justification. In metric space intersection of two open sets is open.
6.	Write whether the following statement is True or False with justification. Continuous image of a compact metric space is compact.
7.	Write whether the following statement is True or False with justification. The closure of a connected set is connected.
8.	Write whether the following statement is True or False with justification.]0,1[and [1,2[are separated.
9.	Write whether the following statement is True or False with justification.]0,1[and]1,2[are separated.
10.	What you can say about connectedness of real line? Justify your answer.
11.	Define Closed sphere.
12.	Define Open sphere.
13.	Define Neighborhood of a point.
14.	Define Closure of metric space.
15.	Define Continuous function on metric space.
16.	Define Convergent sequence.
17.	Define Cauchy sequence.
18.	When the given metric space is said to be complete metric space?
[B]	**3 – Marks Questions**
1.	Define Metric space and give an example of discrete metric space.
2.	Define Pseudo metric space with appropriate example.
3.	Let d(x, y) = min{2,|x-y|}. Show that d is usual metric on R.
4.	For a metric space (X, d), if d*(x, y) = 4d(x, y), then verify whether d*(x, y) is metric space or not.
5.	Let C be the set of all complex number and let $d = C \times C = R$ be defined by $d(z_1, z_2) = \|z_1 - z_2\|, \forall z_1, z_2 \in C$. Prove that (C,d) is a metric space.
6.	Show that every open sphere is an open set but converse is not true.
7.	Let (X, d) be a metric space. If $S \subset X$ then $\bar{S} = S \cup S'$.
8.	Define Convergent sequence in metric space and also show that limit of a sequence, if it exists is unique.
9.	Define Compact metric space and show that every closed subset of a compact metric space is compact.
[C]	**5 – Marks Questions**

1.	Let (X_1, d_1) and (X_2, d_2) be two metric spaces, then for any pair of points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X_1 \times X_2 \to R$ defined by $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ where d_1 and d_2 are metric spaces on X_1 and X_2 respectively, prove that d is metric space on $X_1 \times X_2$.
2.	Let (X, d) be a metric space and ρ be a function on $X \times X$, defined by $\rho(x, y) = \min\{1, d(x, y\} \quad \forall\, x, y \in X$. Then a) (X, ρ) is a bounded metric space. b) ρ is equivalent to d.
3.	Let (X_1, d_1) and (X_2, d_2) be two metric spaces and d^*be the function defined on $X_1 \times X_2$by setting $d^*(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$ then prove that d^* is metric space on $X_1 \times X_2$ such that d^* is equivalent to the product metric space.
4.	Let (X, d) be a metric space and let A and B be arbitrary subsets of X. Then show that a) $A \subseteq B \Longrightarrow \bar{A} \subseteq \bar{B}$ b) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$
5.	Show that the real line is complete metric space.
6.	Every convergent sequence is a Cauchy sequence but converse is not necessarily true.
7.	Define Complete matric space. Show that any set X with the discrete metric forms a complete metric space.
8.	A continuous image of compact metric space is compact.
9.	Let A and B be separated subsets of a metric space (X, d) and let P and Q be non-empty subset of X such that $P \subset A$ and Q $\subset B$. Then P and Q are also separated.
10.	Let (X, d) be metric space and let E be a connected subset of X such that $E \subset A \cup B$ when A and B are separated subsets of X. Prove that either $E \subset A$ or $E \subset B$.
11.	Prove that closure of a connected set is connected.
12.	Let X be a non-empty set and define a mapping d: $X_1 \times X_2 \to R$ as follows $d(x, y) = \begin{cases} 0, when\ x = y \\ 1, when\ x \neq y \end{cases} \quad \forall x, y \in R$ Then show that d is a metric on X.
13.	Let R^n be the set of all ordered pairs of real numbers and let $d: R^n \times R^n \to R$ be defined by $d(x, y) = \{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2\}^{\frac{1}{2}}\ \forall\ x\ , y \in R^n$ where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. Then prove that (R^n, d) is metric space.
14.	Let R^2 be the set of all ordered pairs of real numbers and let $d: R^2 \times R^2 \to R$ be defined by $d(x, y) = \{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{\frac{1}{2}}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then prove that (R^2, d) is metric space.
15.	Let R^2 be the set of all ordered pairs of real numbers and let $d: R^2 \times R^2 \to R$ be defined by $d(x, y) = \max\{[x_1 - y_1], [x_2 - y_2]$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then prove that (R^2, d) is metric space.
16.	Define Open sphere and show that every open sphere is an open set but the converse is not true.
17.	Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A function $f: X \to Y$ is continuous at $a \in X$ iff for each sequence $<a_n>$ in X converging to $a \in X$, the sequence $f(a) \in X$ converges to $f(a)$.
18.	Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A sequence $< (x_n, y_n) >$ in the product space converges to (x,y) iff $x_n \to x$ and $y_n \to y$.
19.	If x and y are any two points is a metric space (X, d) and if $< y_n >$ is a sequence converging to y, then $< d(x_n, y_n) >$ converge to d(x, y).
20.	If $< x_n >$ and $< y_n >$ converge to points x and y respectively ina metric space (X, d) then the sequence $< d(x_n, y_n) >$ converge to d(x,y).
21.	Show that the real line is complete metric space.
Unit-3	**Riemann Integrability**
[A]	**1 – Mark Questions**
1.	Define Partition.

2.	What can be concluded from refinement of a partition?
3.	State Upper Riemann sums.
4.	Write Lower Darboux sums.
5.	If any bounded function f has finite points of discontinuity then whether f in Riemann integrable or not? Justify.
6.	Show that $\int_0^2 x^2 dx^2 = 8$.
7.	Solve $\int_\pi^{2\pi} \sin x \, d(\cos x)$.
8.	Find value of $\int_0^3 \sqrt{x}\ dx^3$.
9.	State formulae for reduction from Riemann-Dtieltjes Integral into Riemann Integral.
10.	Solve $\int_\pi^{2\pi} \sin x \, d(\cos x)$.
[B]	**3 – Marks Questions**
1.	Show that for any bounded function f defined on [a, b] and let m, M be the infimum and supremium of f on [a, b]. Then for any partition P of [a, b], we have $m(b-a) \le L(P,f) \le U(P,f) \le m(b-a)$
2.	Prove that no lower Darboux sum can exceed any Darboux upper sum. Or If P_1 and P_2 are any two partitions of [a,b]. Then $U(P_2, f) \ge L(P_1, f)$.
3.	If f is a bounded function defined on [a, b] and P be and partition of [a, b], then show that $U(P,-f) = -L(P,f)$ and $L(P,-f) = -U(P,f)$.
4.	Briefly discuss upper and lower Riemann integral.
5.	Compute L(P, f) and U(P, f) if f(x)=x2 on [0,1] and P={0, 1/4, 2/4, 3/4, 1} be a partition of [0, 1].
6.	Compute L(P, f) and U(P, f) if f(x)=x on [0,1] and P={0, 1/3, 2/3, 1} be a partition of [0, 1].
7.	Compute L(P, f) and U(P, f) if f(x)=x on [0,1] and P={0, 1/4, 2/4, 3/4, 1} be a partition of [0, 1].
8.	Every continuous function is integrable.
9.	Give an example of a Riemann integrable function on [a, b] which is not monotonic [a, b].
10.	Show that the function f defined as follows: $$f(x) = \frac{1}{2^n}\ when \frac{1}{2^{n+1}} < x < \frac{1}{2^n} : (n = 0, 1, 2 \ldots.), f(0) = 0$$ is integrable in [0, 1], although it has an infinite number of points of discontinuity.
11.	If P_1 and P_2 be any two partitions of [a, b], then $U(P_2, f, \alpha) \ge L(P_1, f, \alpha)$ that is no lower sum can exceed any upper sum.
[C]	**5 – Marks Questions**
1.	Show that a constant function is integrable.
2.	If f(x)=x3 is defined on [0, a], show that $f \in R[a, 0]$ and $\int_0^a f(x)dx = \frac{a^4}{4}$.
3.	Show that f(x)= sin x is integrable on $[0, \frac{\pi}{2}]$ and $\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$.
4.	Show by an example that every bounded function need not be Riemann integrable. Or Let $f(x) = \begin{cases} 0, when\ x\ is\ rational \\ 1, when\ x\ is\ irrational \end{cases}$ Show that f is not integrable on [a, b].
5.	If f(x) be defined on [0, 2] as follows,

	$f(x) = \begin{cases} x + x^2, when\ x\ is\ rational \\ x^2 + x^3, when\ x\ is\ irrational \end{cases}$ then evaluate the upper and lower Riemann integrals of f over [0, 2] and show that f is not R-integrable over [0,2].
6.	Find the upper and lower Riemann integrals for the function f defined on [0, 1] as follows $$f(x) = \begin{cases} (1 - x^2)^{\frac{1}{2}}, if\ x\ is\ rational \\ 1 - x, if\ x\ is\ irrational \end{cases}$$ Hence show that f is not Riemann integrable on [0, 1].
7.	If f(x)=x2 is defined on [0, a], show that $f \in R[a, 0]$ and $\int_0^a f(x)dx = \frac{a^3}{3}$.
8.	If f is defined on [0, 1] by f(x)=x $\forall x \in [0,1]$ then prove that f is Riemann integrable on [0, 1] and $\int_0^1 f(x)dx = \frac{1}{2}$.
9.	A necessary and sufficient condition for the integrability of a bounded function f is, that every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P, whose norm is $\leq \delta$, the oscillatory sum w(P,f) is < ε, i.e. U(P,f)-L(P,f)< ε .
10.	A necessary and sufficient condition for the integrability of a bounded function f is that to every $\varepsilon > 0$ there corresponds a partition P such that corresponding oscillatory sum w(P,f)<ε , i.e. U(P,f)-L(P,f)< ε .
11.	A bounded function f is integrable in [a, b], if the set of its points of discontinuity is finite.
12.	If f is monotonic in [a, b], then it is integrable in [a, b].
13.	If P^* is a refinement of P, then prove that $$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$ or $$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$
14.	Let f and α be bounded functions on [a, b] and α be monotonic increasing on [a, b]. Then the function f is integrable with respect to α on [a, b] if and only if for every $\varepsilon > 0$ there exists a partition P of [a, b] such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.
15.	Let m, M be the infimum and supremum of f on [a, b]. Then, for any partition P of [a, b] $$m\{\alpha(b) - \alpha(a)\} \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\{\alpha(b) - \alpha(a)\}$$
16.	If f is bounded in [a, b] and M, m are the infimum and supremum of f in [a, b], then $$m(b-a) \leq \int_{\underline{a}}^{b} f(x)dx \leq \int_{a}^{\overline{b}} f(x)dx \leq M(b-a)$$
Unit-4	**Measurable function and Lebesgue Integral**
[A]	**1 – Mark Questions**
1.	Define Length of an interval.
2.	Show that if A is countable then $\mu^*(A) = 0$.
3.	Define $\sigma - algebra$.
4.	Is R-Set of real numbers is measurable or not ?
5.	Define Lebesgue measurable set.
6.	Define Lebesgue integral of bounded functions.
7.	When the given function f is said to be Lebesgue integrable on the given set.
[B]	**3 – Marks Questions**

1.	What does $\sum l(I_k)$ mean? For example, suppose $J_1 = I_2, J_2 = I_1, J_3 = I_4, J_4 = I_3$, etc. Is $\sum l(I_k) = \sum l(J_k)$? Does the order make a difference when we sum? Does the labeling matter?
2.	State all properties of Lebesgue outer measure.
3.	Let $A \subset \mathbb{R}$ be an arbitrary set and $\varepsilon > 0$ be given. Then there exists an open set $O \supset A$ such that $\mu^*(O) \leq \mu^*(A) + \varepsilon$. There is a $G_{\delta-set}$ $G \supset A$ such that $\mu^*(G) = \mu^*(A)$.
4.	State properties of Lebesgue outer measure.
5.	Briefly discuss about measurable sets.
6.	Let $E \subset R$, If $\mu^*(E) = 0$ then E is measurable.
7.	Countable union of measurable set is measurable.
8.	Prove that the characteristic function χA is measurable if and only if the set A is measurable.
9.	Let f: D→ R be a measurable function and let $A = \{x \in D: f(x) = 0\}$. If $\frac{1}{f}$ is defined to be α on A for some $\alpha \in R$ then $\frac{1}{f}$ is measurable. If $\mu(A) = 0$ then $\frac{1}{f}$ is measurable irrespective of what values we assign to A.
[C]	**5 – Marks Questions**
1.	Discuss motivation for Lebesgue measure.
2.	Discuss about Lebesgue measurable sets.
3.	Explain Lebesgue outer measure.
4.	If E and F are measurable sets then so are E U F and E-F.
5.	If E1, E2 , . . .,En are n pairwise disjoint measurable sets then for any A⊂ R $$\mu^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} \mu^*(A \cap E_i).$$
6.	If $\{E_n\}_{n\in N}$ is a sequence of pairwise disjoint measurable sets then, i) for any A⊂ R , $\mu^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$. ii) $E = \bigcup_{i=1}^{\infty} E_i is\ measurable.$
7.	Discuss Lebesgue integral for unbounded functions.
8.	State and prove monotone convergence theorem.
9.	Define Lebesgue integral. Let s be a non-negative simple function defined on a measurable set E. Then the function v defined by v(A)=$\int_A s d\mu$ for A⊂E is a measure on m(E), the collection of all measurable subsets of E.
10.	Explain Lebesgue integral.
11.	Prove that Lebesgue outer measureis countably sub-additive.
12.	Prove that Lebesegue outer measure is translation invariant.
13.	Prove that Lebesgue outer measure of an interval is equal to its length.